Lecture 10

Markets, Mechanisms and Machines

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Introduction to game theory

- In introductory microeconomic theory we focus on *price-taking agents*
  - Agents observe prices and make consumption decisions that maximize their utilities under budget constraint
- This models market where agents are small
  - Individual agent action cannot impact the price
- Also, the agents cannot cooperate, collude or compete *directly* with each other
- Note to mechanism designer: properties of markets with interacting agents can be very different from markets with price-taking agents

Introduction to game theory

- Recall that in matching problems the goals of agents and designer are aligned
  - Worst-case scenario: matching is not stable and agents break up matches we created
- When goals of designer and agents do not align, designer needs to provide *incentives* to agents to behave in a way to optimize her objective
- Agents behave selfishly by optimizing their own objectives and responding to incentives of the mechanism and behavior of other agents
- Collective behavior of agents can generate outcomes that are bad for designer's goal if incentives are not set properly

Introduction to game theory

- Game theory studies interaction of utility-maximizing agents
- Utilities of agents directly depend on each other’s actions
  - Examples: Market competition (Target vs. Walmart), competitive games (poker), auctions
- If agent’s utilities depend on each other’s actions, they need to anticipate what their competitors do
- They also need to know if their competitors know that they are trying to anticipate their actions
- In the “steady state” the agents should have an agreement on who has what information as well as a “binding contract” on who makes what action in response to each possible action of her opponents
- Let’s look at these components formally

Notions of game theory

- A game is a collection of agents with their utility functions, action spaces and information sets
- Action profile of the game is the set of all actions taken by all players
- Utility function is a function that for each player takes the entire action profile and outputs a real number (utility) from that profile
- Action space of the player is the set of possible actions that she can take (it may depend on what other players do
- Information set of player contains everything that this player can observe

Notions of game theory

- Depending on what the timing of players' actions is, we can have *simultaneous move* or *sequential move* games
- In sequential move games players make their actions in a sequence
  - Sequence of actions is determined by the rules of the game
  - The entire sequence of actions becomes part of the information set of the game
  - Examples?
- In simultaneous move games all players make their actions simultaneously
  - There is no way for players to respond to actual actions of their opponents
  - Opponent actions are NOT in the information set
  - Examples?

Complete information games

- From now on we focus on simultaneous move games (I omit "simultaneous move" label moving forward)
- A complete information game is the game in which players information sets include action spaces and utility functions of all their opponents
- Formally, a complete information game is $G=\{N, \{A_i, u_i\}_{i \in N}\}$
  - N is the set of players
  - A_i is player i's action space
    - The entire action profile a is drawn from product set of A_i's
    - For convenience we denote a_{-i} the action profile of all players excluding i
  - u_i is player i's utility mapping a into real numbers

Complete information game

- A strategy of player *i* is her action that she chooses after observing the information set
- The player can choose a fixed strategy or she can randomize over her action space A_i
- Randomized strategies are called mixed strategies
- Formally, a mixed strategy is a probability distribution over A_i
- Player choose action by independently randomly drawing from that distribution

Complete information games

Definition: A (pure strategy) Nash equilibrium of game G is a strategy profile **a** such that for each player i: $u_i(\boldsymbol{a}) \geq u_i(a_i', a_{-i})$ for all $a_i' \in A_i$

- Pure strategy Nash equilibrium is an action profile so that no agent has an incentive to deviate from it
- There could be many pure strategy Nash equilibria or none at all

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Definition: A (mixed strategy) Nash equilibrium of game G is a distribution profile $\{f_i\}_{i \in N}$ over A_i such that for each player i:
$\int u_i(\boldsymbol{a}) f_1(a_1) \ldots f_i(a_1) \ldots f_N(a_N) \mathrm{d}a_1 \ldots \mathrm{d}a_N \geq$
$\int u_i(\boldsymbol{a}) f_1(a_1) \ldots f_i'(a_1) \ldots f_N(a_N) \mathrm{d}a_1 \ldots \mathrm{d}a_N$ for all distributions f_i' over A_i

- Mixed strategy Nash equilibrium always exists

Prisoner’s dilemma

		Prisoner 2	
		Stay silent	Testify
Prisoner 1	Stay silent	**-1,-1**	**-10,0**
	Testify	0,-10	-5,-5

Battle of the sexes

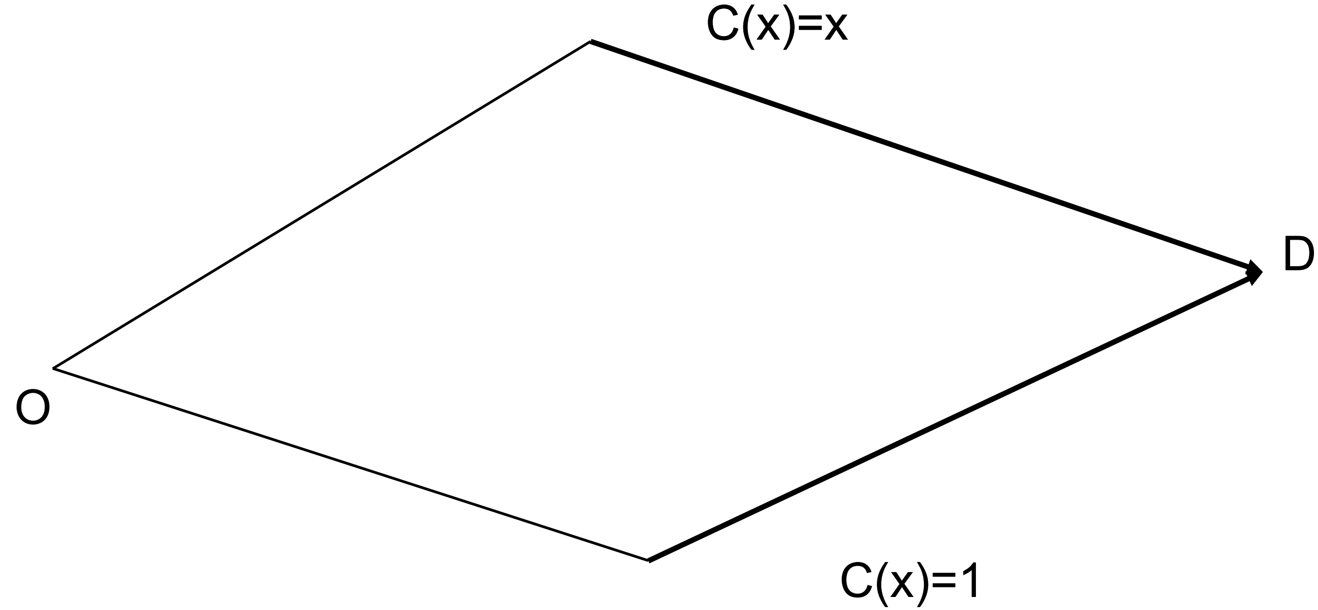
		Player 2	
		Ballet	Boxing
Player 1	Ballet	**2,1**	**0,0**
	Boxing	0,0	1,2

Matching pennies

		Player 2	
		Heads	Tails
Player 1	Heads	**1,-1**	**-1, 1**
	Tails	-1, 1	1, -1

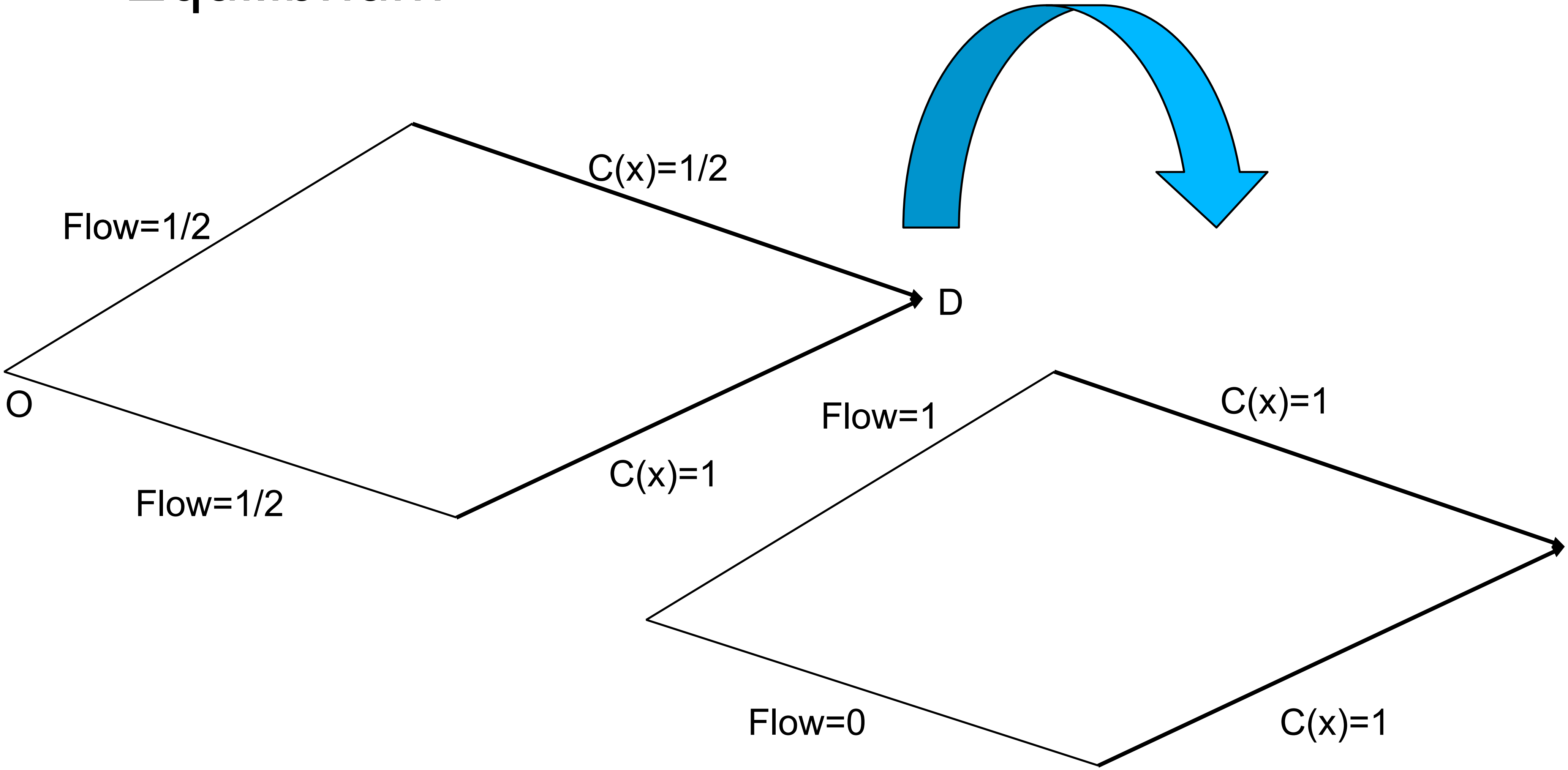
Selfish routing

- “Small” agents (relative to overall volume)
- Each agent wants to optimize path
- Traffic flow is at Nash equilibrium if it based on minimum cost paths
- Total cost of flow is equal to the sum of costs of all agents

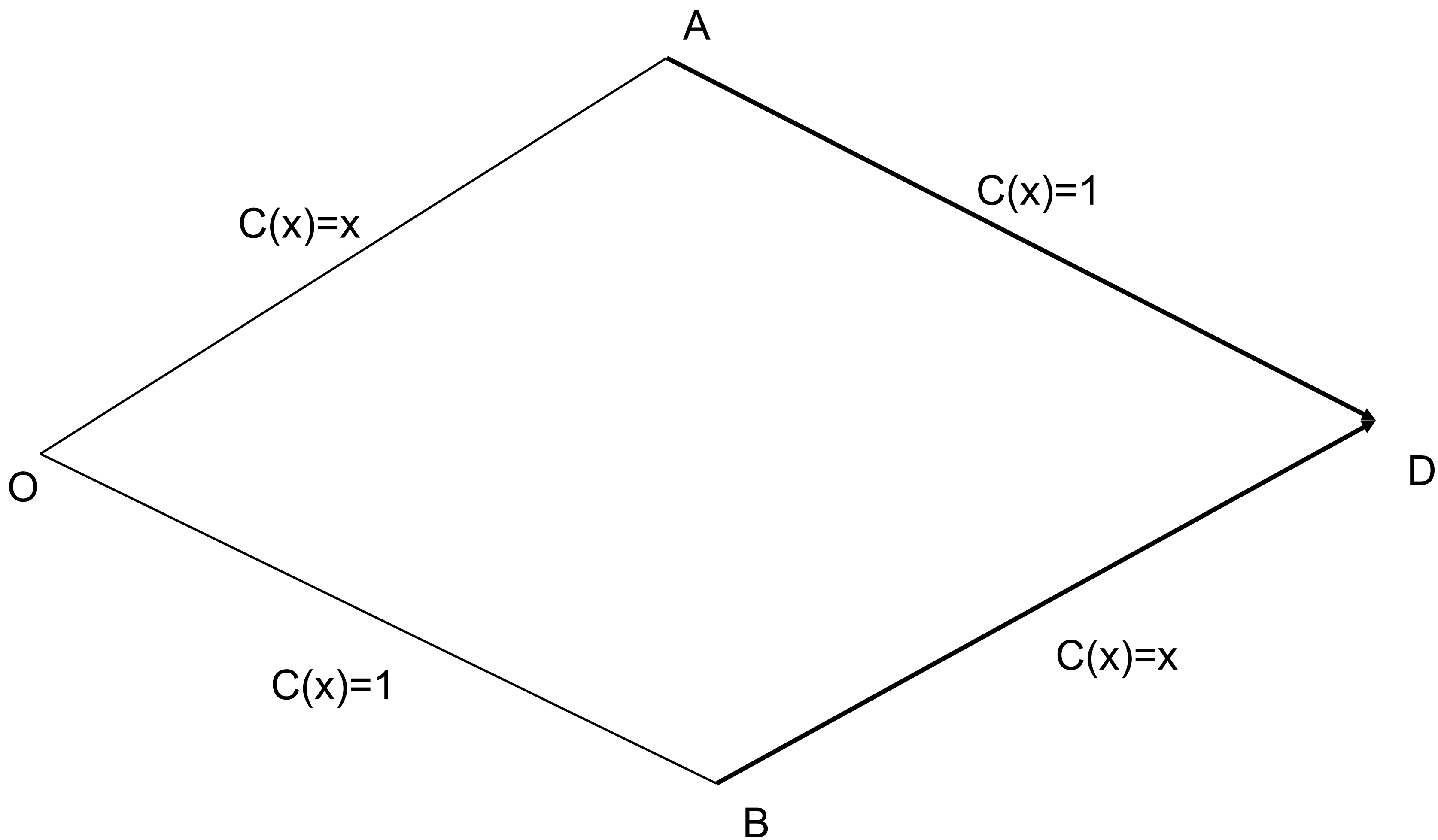


Selfish routing

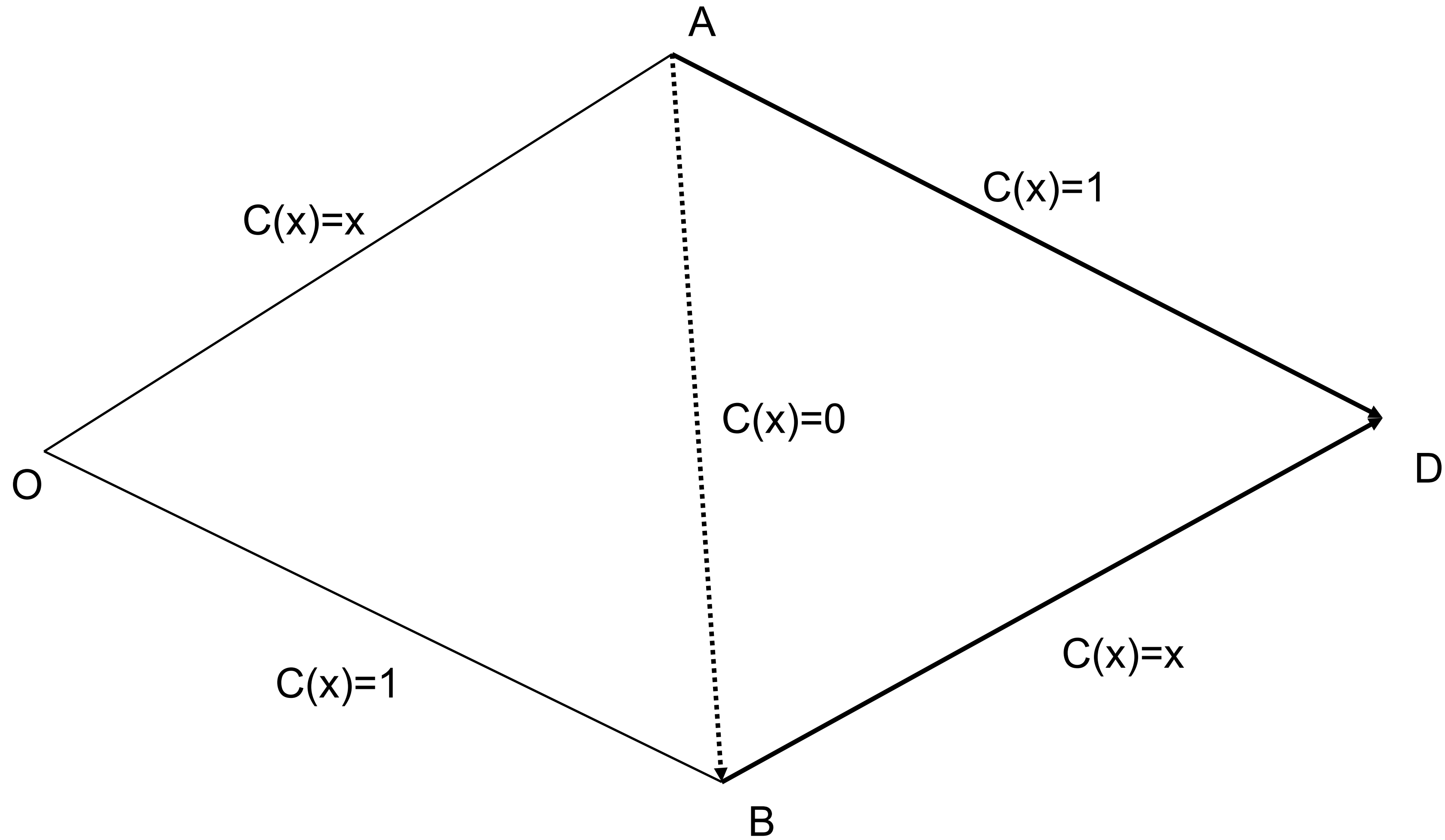
- Equilibrium



Selfish routing



Selfish routing



Selfish routing

Incomplete information games

- In reality, assuming complete information may not be possible
- Players may have information that they do not want to (or cannot) share with other players or the designer of the game
  - Cost of manufacturing or acquisition
  - Maximum willingness to pay in an auction
  - Cognitive ability in collaborative classroom
- We need to adjust the structure of the game to allow players to have private information

Incomplete information games

- To have a self-contained game structure we need to have a model for how players acquire and use private information
- We model private information for each player *i* by a scalar or vector τ_i. We call τ_i the type (or signal) of player *i*.
- We assume that another player (Nature) assigns types to players by drawing them from distribution *D*
- As soon as each player learns her type, she conceals it and does not reveal it to other players
- It, however, remains in her information set

Incomplete information games

- Formally, the incomplete information game is $\Gamma=\{N,\{A_i,T_i,u_i\}_{i\in N},D\}$
  - Set of players N
  - Action space A_i for player i
  - Type space T_i that contains possible values of this player's type τ_i
  - Distribution over types D
  - Utility function u_i that maps action profile and type of the player into real numbers

Incomplete information games

- In complete information game if players agree to choose a specific (equilibrium) action profile, no player has an incentive to deviate from it because it decreases utility
- AND all players know it
- In incomplete information game each player can *claim* that deviation from an agreed upon action profile does not decrease her utility
- AND other players cannot verify this claim

Incomplete information games

- So the game leads to actions motivated by players' types
  - Example: Assume that with some low probability Player 1 in the Battle of the Sexes can have utility of -10 from boxing
- If a player knows D then if her opponent makes a claim about her type, you can judge if that claim is likely true
- Then each player needs to construct a probability distribution that would reflect the likely actions of their opponents

Incomplete information games

- We call this beliefs of players
- Formally, these are conditional distributions

  $D(\tau_{-i} \mid \tau_i)$
- Knowing their own type, each player tries to predict the types of all their opponents
  - Just like in the card game each player tries to predict the cards of everyone else knowing their own cards

Incomplete information games

- Only τ_i is in the information set of each player
- The strategy of the player prescribes action that correspond to given information
- In the incomplete information game the strategy of player i is the mapping β_i from T_i into A_i
- Once the player learns her type, she makes an action
- Unlike complete information games we focus only on deterministic such functions
- The strategy profile of the incomplete information game is $\beta(\tau)=(\beta_1(\tau_1),\ldots,\beta_N(\tau_N))$

Incomplete information games

- We define *ex post* utilities of players (i.e. when the uncertainty of types was revealed after the game was played)

  $u_i(\beta(\tau), \tau_i)$

- *Interim* utilities of players (i.e. when player learns her type but does not know the types of others)

  $E[u_i(\beta(\tau), \tau_i) | \tau_i]$

- Ex ante utilities of players (i.e before nature assigns types)

  $E[u_i(\beta(\tau), \tau_i)]$

Incomplete information games

- Strategy β_i weakly dominates β_i‘ if for any a_{-i} and τ: $u_i(\beta_i(\tau_i), a_{-i}, \tau_i) \geq u_i(\beta_i'(\tau_i), a_{-i}, \tau_i)$
  - The inequality is strict for some of those alternative strategies
- Strategy β_i is dominant if it dominates any other strategy β_i‘
- Strategy β_i is undominated if no strategy dominates it

Incomplete information games

Definition: A dominant strategy equilibrium of an incomplete information game Γ is a strategy profile β_i such that for any $i \in N$ and $\tau \in T$ and $a_{-i} \in A_{-i}$ and any strategy β_i‘:
$u_i(\beta_i(\tau_i), a_{-i}, \tau_i) \geq u_i(\beta_i'(\tau_i), a_{-i}, \tau_i)$

Definition: A Bayes-Nash equilibrium of an incomplete information game Γ is a strategy profile β_i such that for any $i \in N$ and $\tau_i \in T_i$ satisfies:
$E[u_i(\beta(\tau), \tau_i) | \tau_i] \geq E[u_i(a_i, \beta_{-i}(\tau_{-i}), \tau_i) | \tau_i]$ for any $a_i \in A_i$

- If a given strategy profile is a dominant strategy equilibrium, it is also BNE

Incomplete information games

- Properties of BNE:
  1. Strategy of each player is interim-optimal
  2. Strategy of each player is ex ante optimal
  3. We can define ex post equilibrium as an equilibrium where after observing each other's types players do not want to deviate from the BNE profile (important for settings with repeatedly interacting players)
  4. Ex post equilibria are a subset of BNE and BNE is a subset of ex ante equilibria

Example

- Buyer and seller want to trade an object
- Buyer's value for object is \$3
- Seller's value is either \$0 or \$2 depending on type {L,H}
- Buyer can offer either \$1 or \$3 for the object
- Seller chooses whether to sell or not

Example

Seller's type =L

	sale	no sale
\$3	0,3	0,0
\$1	2,1	0,0

Seller's type =H

	sale	no sale
\$3	0,3	0,2
\$1	2,1	0,2

- Regardless of type distribution, this game has BNE where $\beta_S(L)$=sale, $\beta_S(H)$=no sale, and β_B=\$1
- Selling is weakly dominant when seller has type L
- Offering \$1 is weakly dominant for buyer