Lecture 8

Markets, Mechanisms and Machines

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Centralized resource allocation

- Fundamental problem in Economics
  - Central planner optimizing objective function (e.g. social welfare)
  - Scarce resource endowment
  - Production technology
  - Consumption choice
- Very important: centralized allocations correspond to most optimal solution (“first best” if directions of central planner are mandatory)
  - Decentralized solutions are compared to the centralized one and sometimes can have similar properties (Arrow-Debreu)

Centralized resource allocation

Large scale attempt

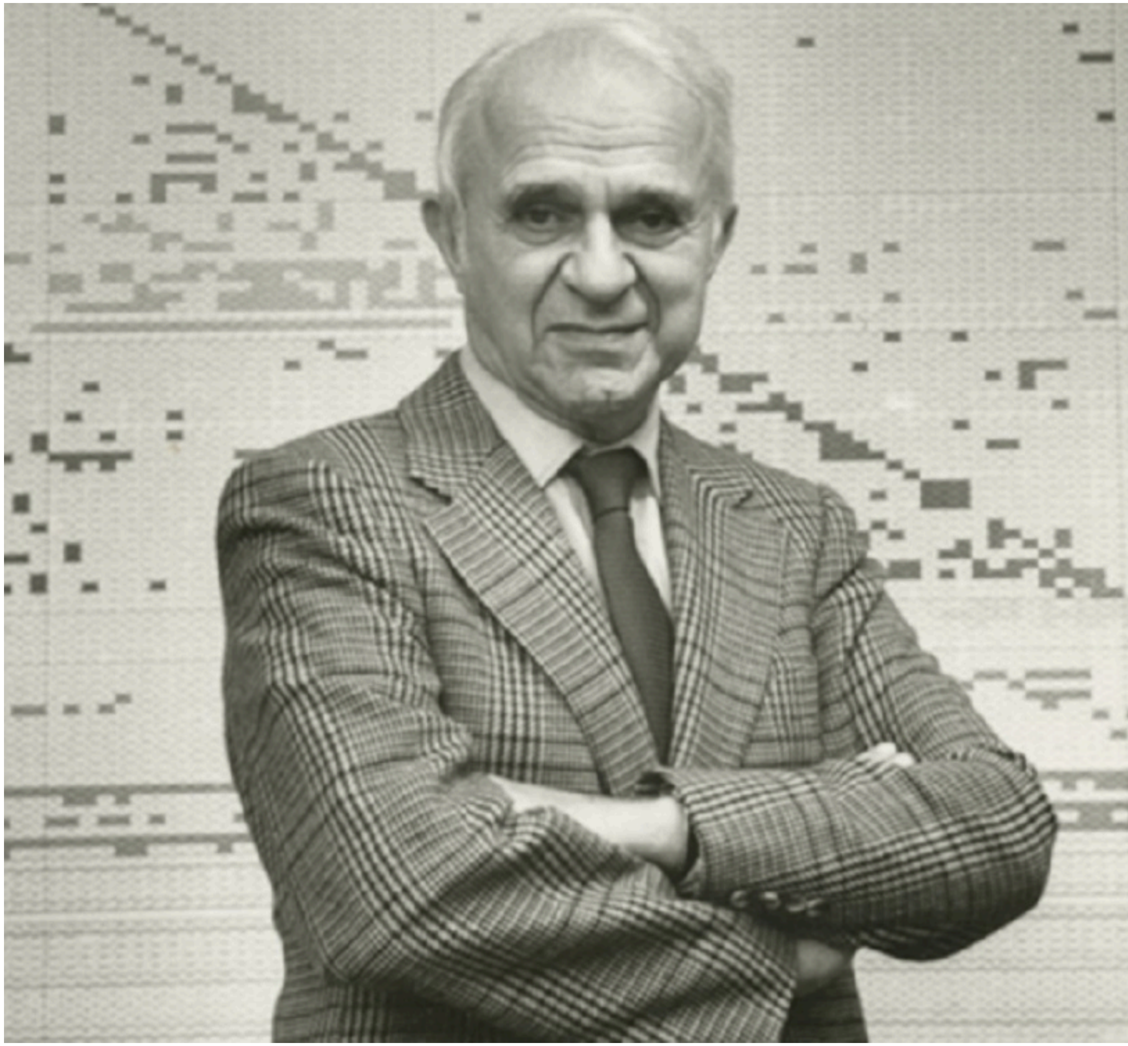
Soviet “Gosplan” (State Planning Committee)

Input-output tables

- Idea by Wassiliy Leontief

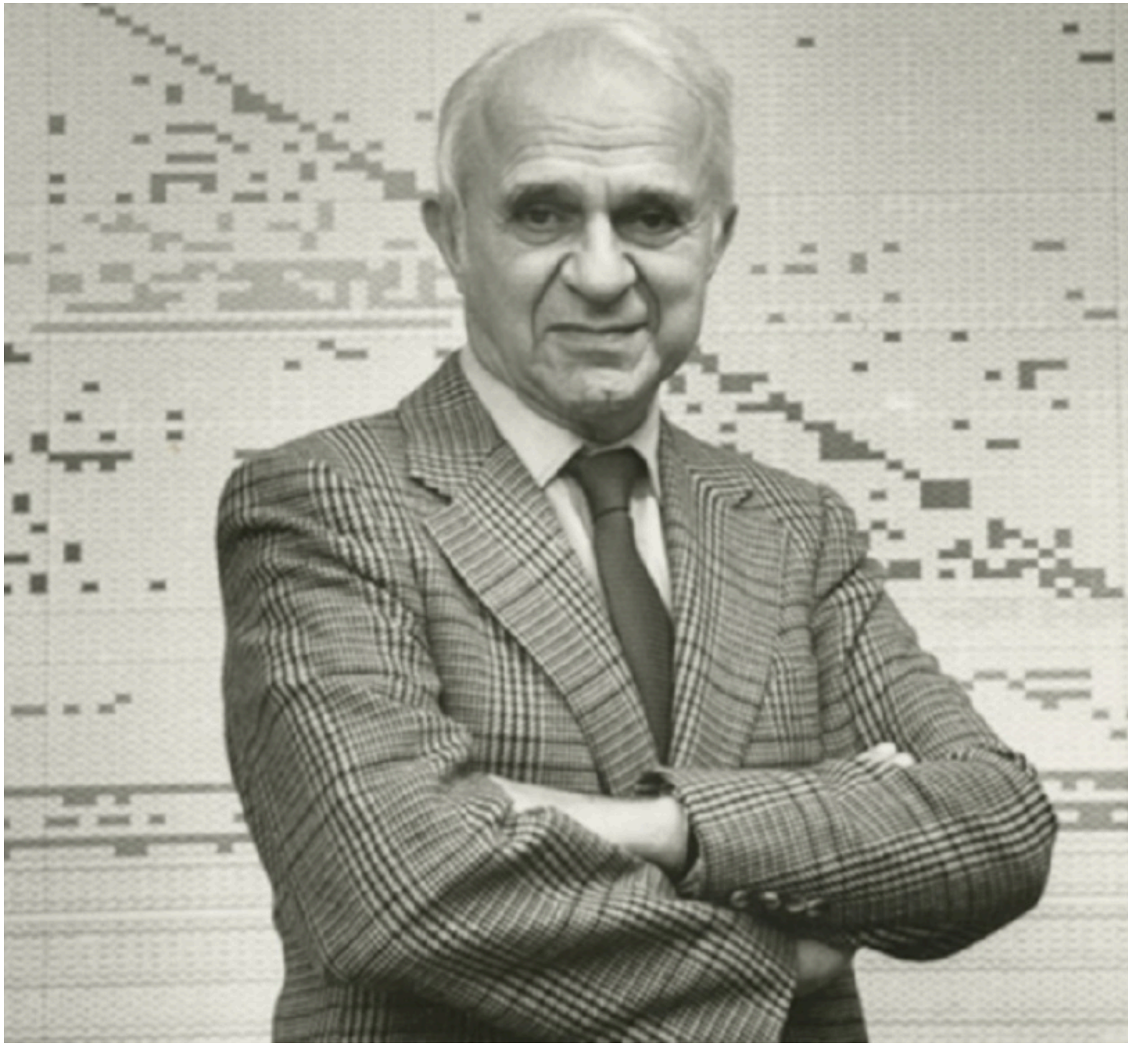
(Nobel Prize, 1973)

- Leontief production function



Input-output tables

- Idea by Wassiliy Leontief (Nobel Prize, 1973)
- Leontief production function
  - Vector of inputs x
  - Used in fixed proportions
  - Matrix A contains proportions of consumption of all inputs x to produce all possible outputs
  - A·x is vector of all outputs from inputs x
  - A forms input-output table



Market design

- Choose how to allocate production across production facilities to produce *planned* amounts of final outputs
  - To minimize production cost
- This is a very large matching problem!
  - Match production facilities with corresponding inputs and production plan

Market design

- Mathematically we need to solve the problem

$$\min_x \sum_i c_i\, x_i$$

subject to $\sum_j a_{ji} x_i \geq b_j$, $j=1,\ldots,N$.

- Here a_{ij} are elements of matrix A and b_j are planned outputs of each final product
- How does microeconomic theory call this problem?

Linear programming

- $\min_x \sum_i c_i x_i$

  s.t. $\sum_j a_{ji} x_i \geq b_j$

is a linear programming problem

- Idea by Leonid Kantorovich

(Nobel Prize, 1975)

- Turns out to be fundamental

for optimization

- Many problems are solved

via a reduction to LP

Linear programming

- Canonical form
- $\max_x c^T x$
- $Ax = b,\ x \geq 0$
- Normal form
- $\max_x c^T x$
- $Ax \leq b,\ x \geq 0$
- Both forms are equivalent (by construction of extra “slack” variables that turn inequalities into equalities

Linear programming

- The key question in linear programming is the characterization of the *feasible set*
- $\{x : Ax = b,\ x \geq 0\}$
- This is a convex polytope
- Recall: set S is convex if for all x,y in S and $0 \leq \lambda \leq 1$, $\lambda x + (1-\lambda)y$ is in S

Linear programming

- Recall: x is an extremal point of a convex set if it cannot be represented as a weighted combination of any of its other points
- Minkowsky’s theorem: A convex compact set is a linear envelope of its extremal points
- Idea: to characterize a convex polytope it is sufficient to list all its vertices (which are its extremal points)

Simplex algorithm

- (Informally stated) theorem: The objective attains its maximum at one or more of extreme points of the convex polytope $\{x : Ax = b, x \geq 0\}$
- The algorithm then is
  1. Start at the initial extreme point
  2. Verify if the objective increases along any edges of the polytope
  3. Follow the edge of maximum increase of the objective to the next extreme point
  4. Stop when objective no longer increases

Linear programming

- Simplex method is implemented in all optimization packages
- However
  - There is no guarantee that it “generically” converges
  - The worst case LP takes exponential time to solve with the simplex method
- Nevertheless, simplex method is still very popular
  - George B. Danzig (developed the simplex algorithm): “The simplex method is still running”
- Can LPs be solved efficiently?

From LP to feasibility

- Minimization problem of LP is replaced with the feasibility problem
- This can be done, for instance, by considering set

$P=\{c^T x \geq d, Ax = b, x \geq 0\}$

for some large d (that can be adjusted up or down as the algorithm runs)

From LP to feasibility

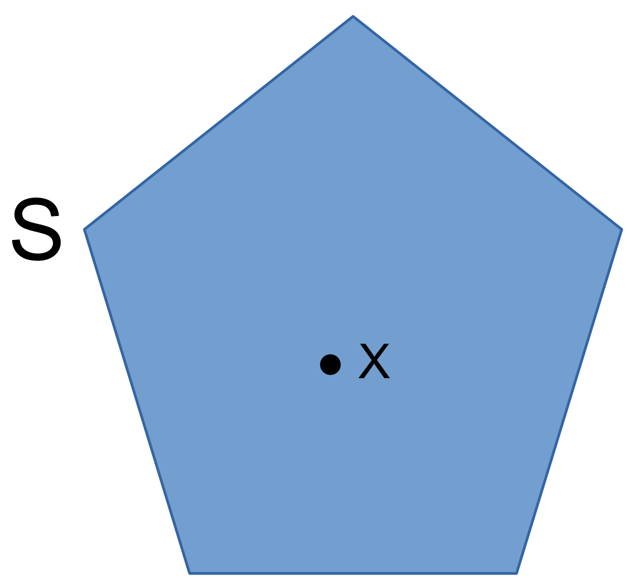
- Select $P=\{c^T x \geq d, Ax = b, x \geq 0\}$

for some appropriate d

- Then solving the LP reduces to finding (any) point x in P
- In most settings we can find a convex set that is contain solution but how do we approach the solution?

Separation oracles

- For convex set S a polynomial time separation oracle is an algorithm such that for each point x it states that x is in S or it produces a hyperplane $a^T x=e$ that separates S from x



Oracle: x is in S

Separation oracles

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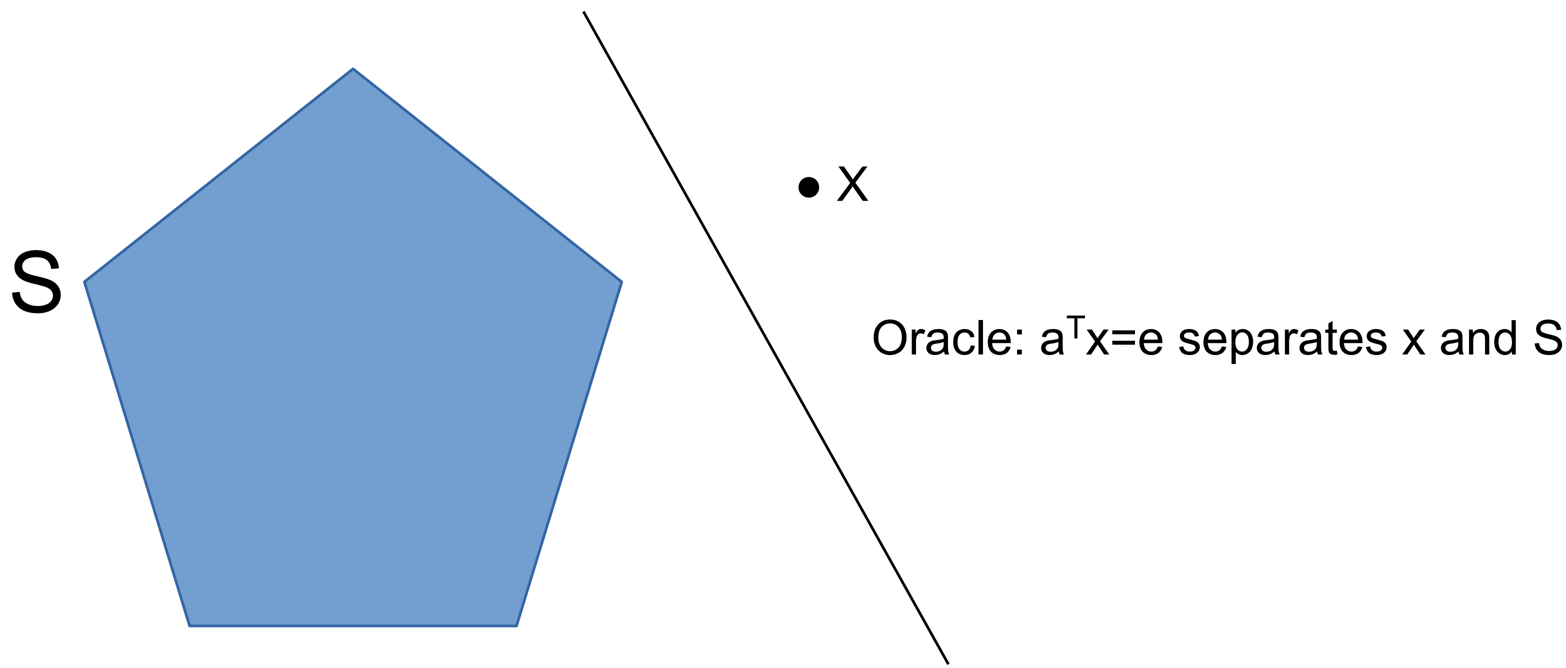
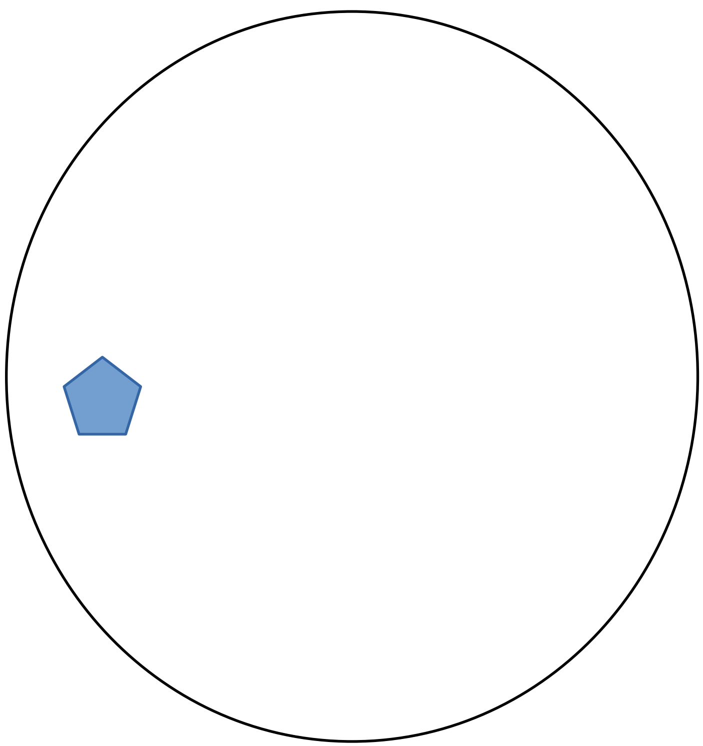


Ellipsoid method

- Goal: find feasible point of a convex set S
- Ellipsoids: affine transformations of spheres
  - Example: $a^2x^2+b^2y^2=c^2$
- Algorithm
- 1. Initialize by finding an ellipsoid that contains S (typically a sphere)



Ellipsoid method

- Algorithm
- 1. Initialize by finding an ellipsoid E_0 that contains S (typically a sphere)
- 2. Take the center of E_0 x_0 and call separation oracle for S at x
- - If x is in S, feasible point found
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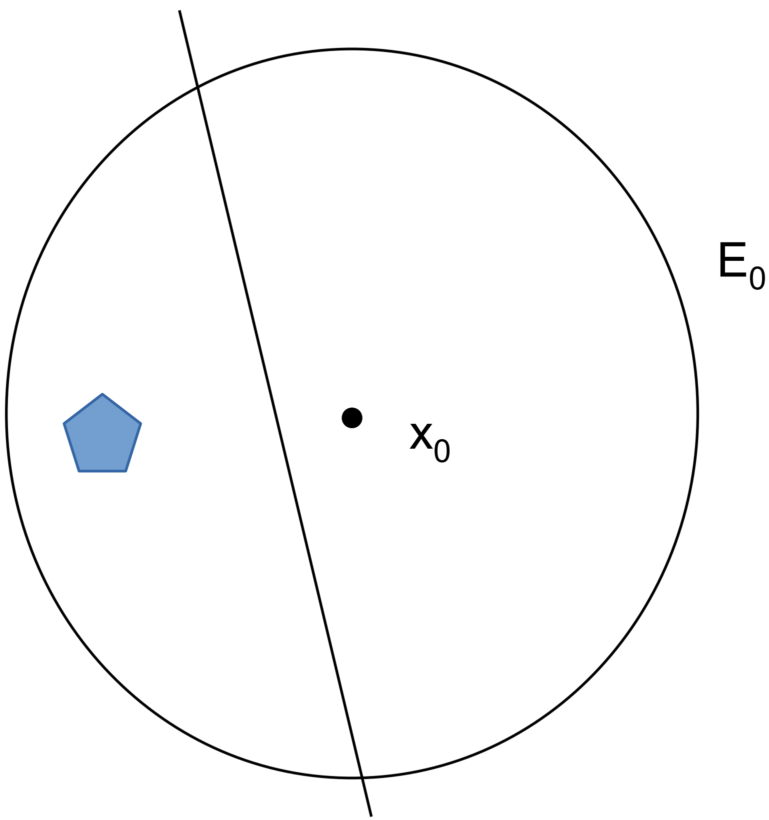

Ellipsoid method

- Algorithm
- 3. Otherwise, take the slice of E_0 cut by the hyperplane produced by the separating oracle and construct the smallest ellipsoid E_1 that contains that slice
- 4. Move to the center of the new ellipsoid and call the separating oracle there
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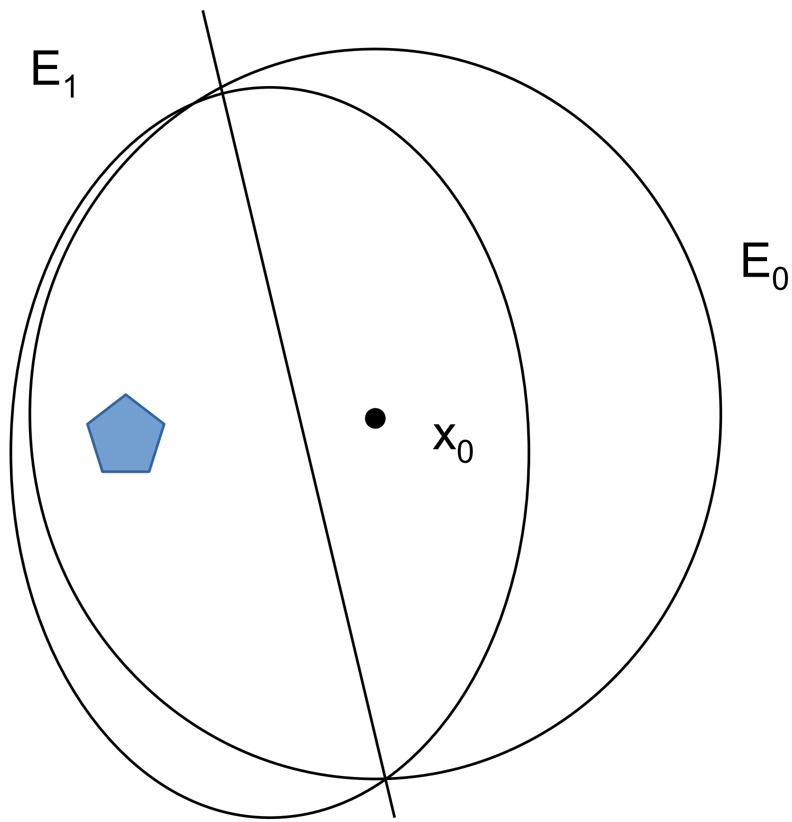

Ellipsoid method

- The algorithm terminates if the solution is found or the scale of the ellipsoid is smaller than pre-selected tolerance ε
- What it the number of steps of ellipsoid algorithm that is required to reach accuracy ε?
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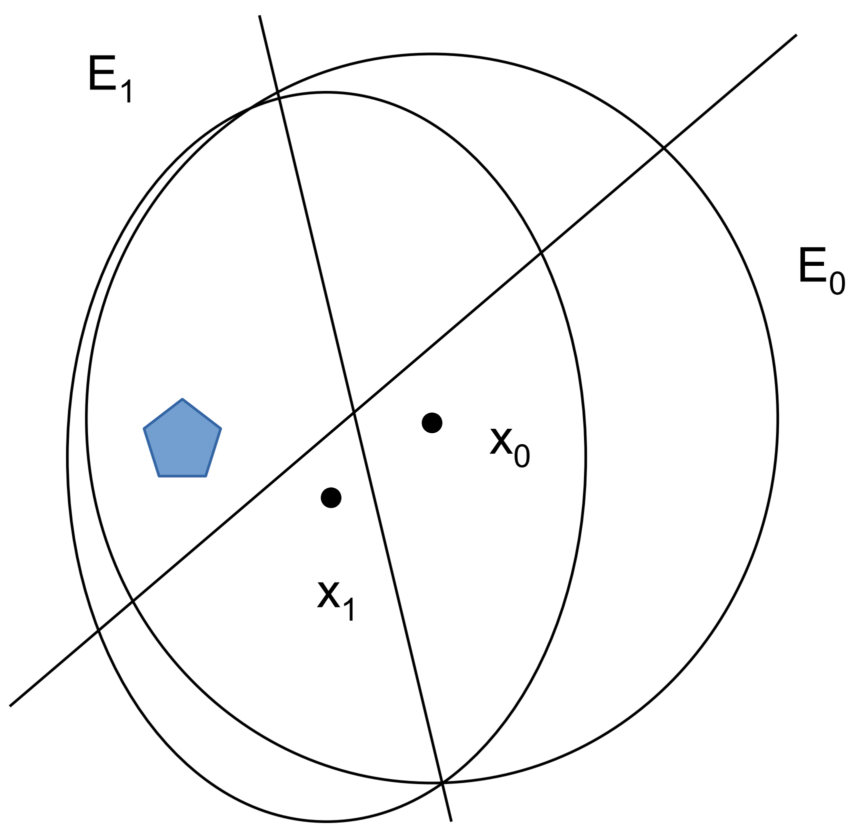


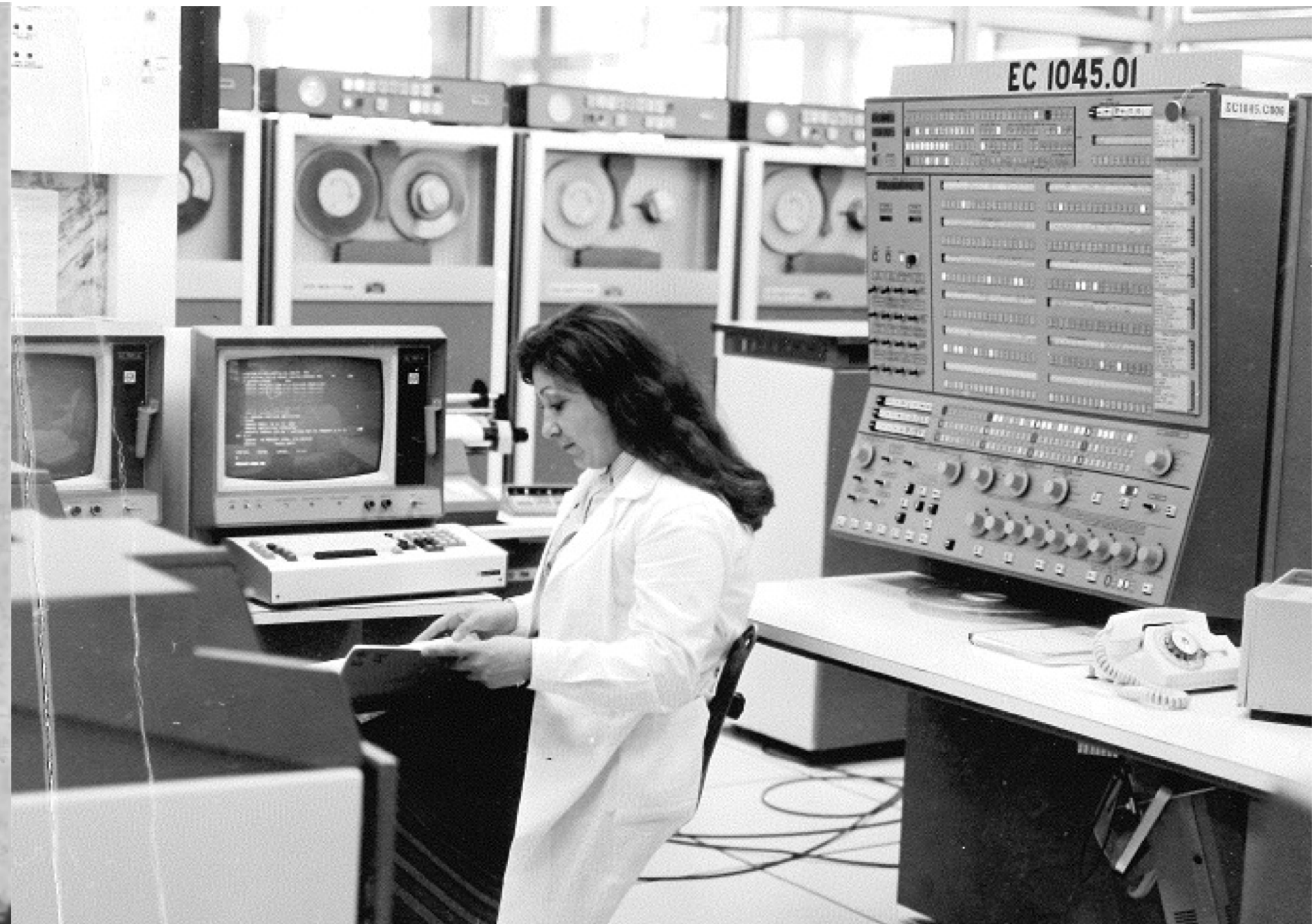
Slicing a convex cake

- Problem: you and your friend baked a convex cake (not necessarily symmetric). You decided to split the cake using the following rule:
  - You pick the point on the cake
  - Your friend makes a single straight cut and you get the part of the cake that contains your point
- How do you pick the point?
- Easy for 1-dimensional cakes

Slicing a convex cake

- Much more difficult for 2 or higher-dimensional cakes
- Answer: centroid (center of mass) of the cake
- Grunbaum’s theorem: A cut through the centroid of a convex set S produces the subsets with the volume of at least 1/e of the volume of S
- This gives a rough evaluation of convergence for ellipsoid method:
  - The volume of the slice produced by the separation oracle is at most 1-1/e of the original volume

Centralized resource allocation



- Main computing center of ”Gosplan”
- Computes solution to matching problem
- Directives are mandatory for producers

Allocation of scarce resources

- Gosplan implemented matching in directive way because decentralized implementation requires prices
  - E.g. Recall First Welfare theorem
- Using prices may not be feasible in some existing markets
  - Difficult to set prices (dating?)
  - Unfair to set prices (school matching)
  - Unethical to set prices (organ transplants)

Kidney transplantation

- Required as a result of several kidney diseases
- Current waitlist in the US 102,893
- In 2018 3,854 patients died while waiting
- 14,725 patients received transplant from a deceased donor
- 6,433 patients received transplant from a living donor

Kidney transplantation

- Using prices is not just unethical, it is illegal in the US

PUBLIC LAW 98–507—OCT. 19, 1984 98 STAT. 2339

Public Law 98–507
98th Congress

An Act

To provide for the establishment of the Task Force on Organ Transplantation and the Organ Procurement and Transplantation Network, to authorize financial assistance for organ procurement organizations, and for other purposes.

Oct. 19, 1984
[S. 2048]

Be it enacted by the Senate and House of Representatives of the United States of America in Congress assembled, That this Act may be cited as the "National Organ Transplant Act".

National Organ Transplant Act.
42 USC 201 note.
Health.

TITLE I—TASK FORCE ON ORGAN PROCUREMENT AND TRANSPLANTATION

Kidney transplantation

- Using prices is not just unethical, it is illegal in the US

TITLE III—PROHIBITION OF ORGAN PURCHASES

Penalties.
42 USC 274e.

SEC. 301. (a) It shall be unlawful for any person to knowingly acquire, receive, or otherwise transfer any human organ for valuable consideration for use in human transplantation if the transfer affects interstate commerce.

(b) Any person who violates subsection (a) shall be fined not more than $50,000 or imprisoned not more than five years, or both.

(c) For purposes of subsection (a):

(1) The term "human organ" means the human kidney, liver, heart, lung, pancreas, bone marrow, cornea, eye, bone, and skin,

Kidney exchange

- Prior to December 2004 only 5 exchanges had been accomplished at the fourteen transplant centers in New England.
- Some exchanges had also been accomplished at Johns Hopkins in Baltimore, and among transplant centers in Ohio.
- Why had so very few happened?

Kidney exchange

- Prior to December 2004 only 5 exchanges had been accomplished at the fourteen transplant centers in New England.
- Some exchanges had also been accomplished at Johns Hopkins in Baltimore, and among transplant centers in Ohio.
- Why had so very few happened?
  - Thickness of the market: incompatible donors were simply rejected
  - Due to medical privacy restriction potential donor’s information was not passed on

Kidney exchange

- Roth, Sönmez, and Ünver (2004): demonstrate that number of transplants significantly increases from the appropriately designed exchange
  - Form database of incompatible patient-donor pairs
  - An exchange is a cycle of incompatible patient-donor pairs that points from one incompatible pair to another one, but it which the patient is compatible with the donor in the first pair
  - No restriction on the length of cycles
- The model for the New England Program for Kidney Exchange is based on the model from Roth, Sönmez, and Ünver (2004)

Kidney donation

- Deceased donor’s kidney is allocated via a centralized mechanism
- Living donor is a friend or family member
- For a successful transplant, the donor kidney needs to be compatible with the patient
    - Blood type
    - Protein compatibility
- If the donor and patient are not compatible, transplant is impossible

Donor compatibility

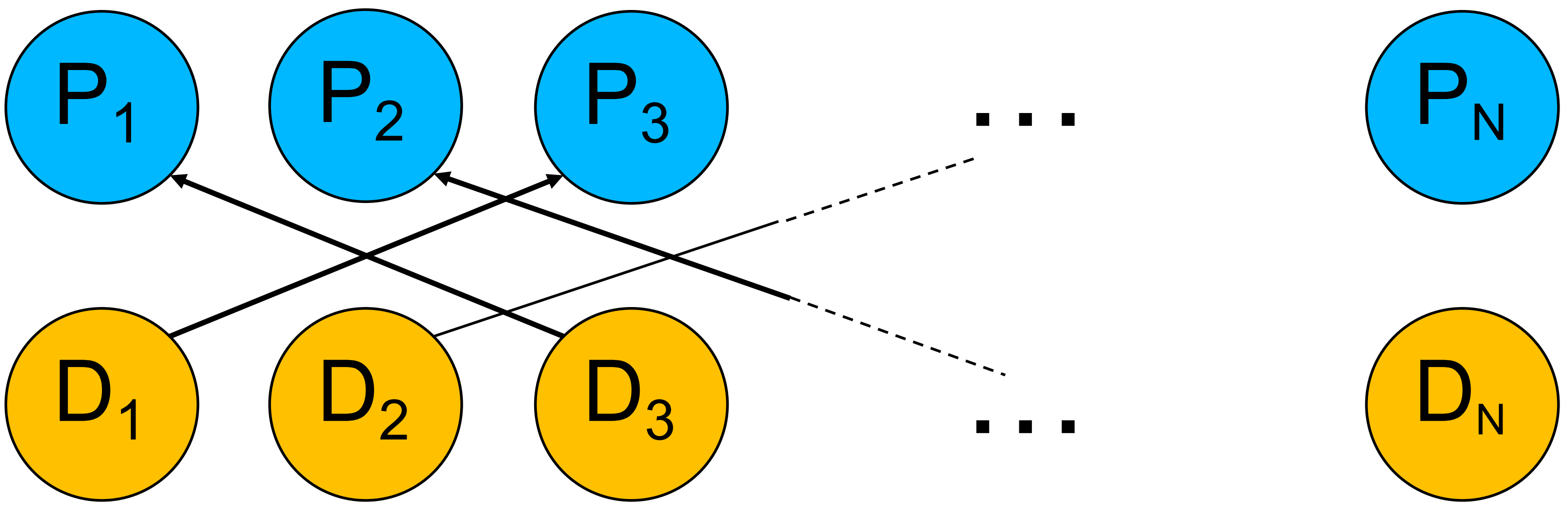
- Blood type compatibility: Blood types, O, A, B and AB.
  - O type patients can receive kidneys from O type donors
  - A type patients can receive kidneys from O or A type donors
  - B type patients can receive kidneys from O or B type donors
  - AB type patients can receive kidneys from donors of any blood type
- Human leukocyte antigen compatibility determines if the immune system of the donor is compatible with the immune system of the patient (otherwise, transplant will be rejected)

Kidney exchange

- **List exchange:** patient is matched with the list of
- **Paired exchange**: Match two patient-donor pairs where donor and patient in each pair are incompatible but the match creates two compatible patient-donor pairs
- The donor of the first pair donates kidney to the patient in the second pair and the donor in the second pair donates kidney in the first pair

Paired exchanges

- Paired exchanges are the most important with living donors



- Mathematically, the problem is finding cycles on graphs (length of cycle is typically bounded)

Kidney exchange

- For incentive reasons, exchanges done simultaneously
- Long cycles require a lot of operating rooms (2 for each pair)
- Original algorithm focused only on two-way exchanges
- Roth, Sönmez and Ünver (2007) showed that allowing 3-way exchanges leads to the nearly optimal number of matches

Kidney exchange

- Further improvement of thickness of the market possible when list exchange is integrated with the paired exchange
- Non-directional donor is living donor willing to donate kidney without having a particular patient in mind
- Such donors before were matched with patients on the list exchange

Kidney exchange

- Integration of non-directional donor to the paired exchange starts a chain where a non-directional donor is matched with the compatible patient in the pair and that pair’s donor is further matched with a patient on another pair
- In conventional exchange if the exchange breaks after some pair has donated a kidney (but before receiving it), the pair cannot make another exchange
- With a non-directional donor, this situation is impossible
  - Chain exchanges can be execute sequentially, avoiding congestion problems

Resource allocation problem

- k-cycle: a cycle over k vertices in the graph such that each candidate obtains the organ of the neighboring donor
- If there are non-directional donors, we can also have chains
- Resource allocation problem
  - Find partition of compatibility graph into chains and cycles with a given bound on length
  - Which optimizes a given criterion (how do we choose it?)

Resource allocation problem

- Let x_{ij} is binary variable equal to 1 if i is matched with j
- Optimal allocation solves

$$\max_x \sum c_{ij} x_{ij}$$

Subject to

$$\sum_j x_{ij} = \sum_j x_{ji}$$

$$\sum x_{ij} \leq 1$$

+ the length constraint

Resource allocation problem

- Note that this has the same structure as Gosplan’s problem
  - Production constraint transforms into “feasibility” constraint (i.e. which pair’s donor is compatible with which pair’s patient, each patient needs only one donor)
  - Function that is being optimized is a measure of “social welfare”
- But this problem is discrete, continuous solution corresponding to LP would not work
- This is integer programming problem

Resource allocation problem

- The solution is based on *relaxation* of the integer program

$\max_x \sum c_{ij} x_{ij}$

Subject to

$\sum_j x_{ij} = \sum_j x_{ji}$

$\sum x_{ij} \leq 1$

+ the length constraint

- A sequence of LPs approximate the solution of the IP

Kidney exchange

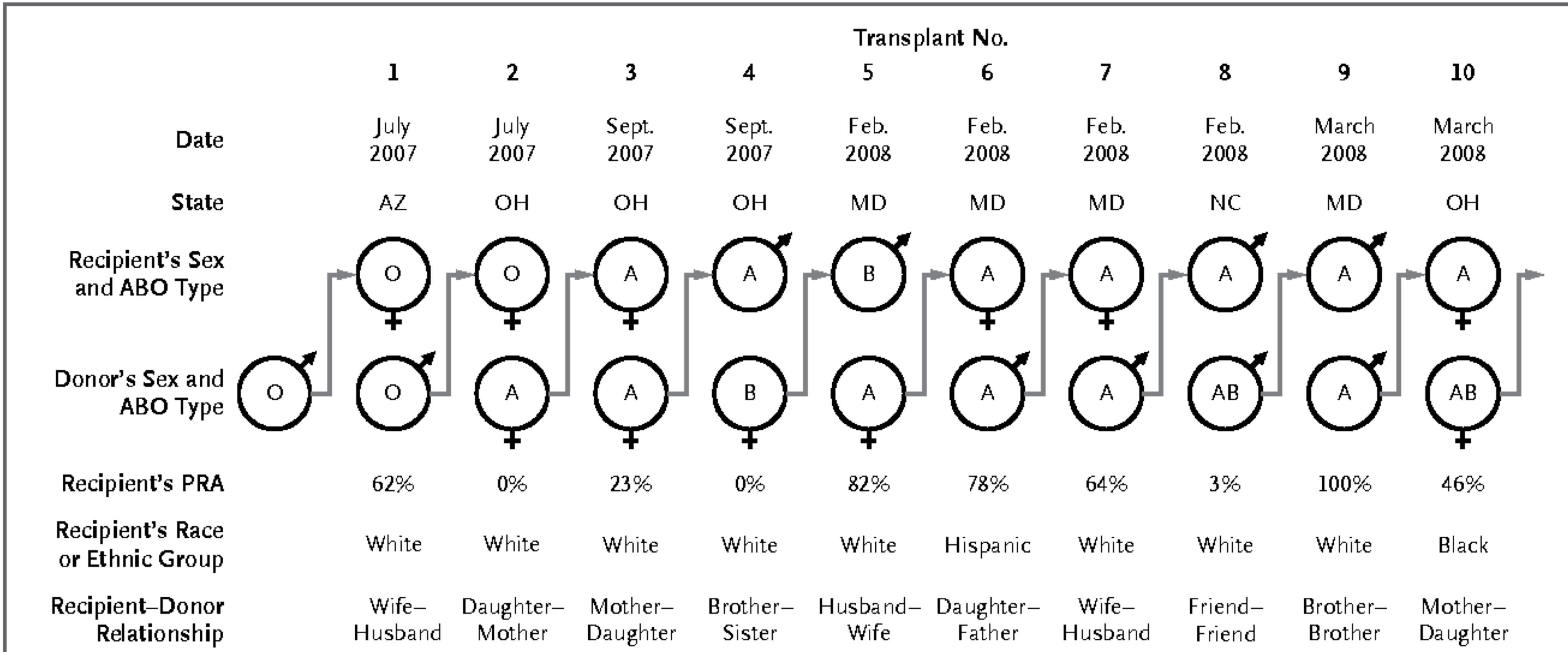


Figure 1. A Nonsimultaneous, Extended, Altruistic-Donor Chain.

In less than 1 year, 10 patients (5 with panel-reactive antibodies [PRAs] >60%) were given a transplant; an 11th transplantation is possible. The initiating donor was an unpaired altruistic donor from Michigan. To date, none of the bridge donors have reneged. The recipient of transplant 6 required desensitization of HLA donor-specific antibodies by T-cell and B-cell flow cytometry. The recipient of transplant 9 had an anti-B antibody titer of 1:8 (as assessed with the use of an antihuman globulin reagent) and required desensitization to blood group.

"A Nonsimultaneous, Extended, Altruistic-Donor Chain," by M. Agnes van Rees, Jonathan E. Kopke, Ronald P Pelletier, Dorry L Segev, Matthew E. Rutter, Alfredo J. Fabrega, J